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MEMORANDUM
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SOME LIMIT THEOREMS ASSOCIATED WITH A RECURRENT EVENT

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SUPPLEMENT TO RM-3491-PR

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The following is intended to correct some errors in the above Memorandum and to add a few comments which seem appropriate. We start with errors.

Page 6, line 12 should be:

$$\lim_{t \to 1^-} \left| \sum_{k=1}^{\infty} \frac{1 - \Delta_k}{k} \right| < \infty .$$

Page 17, line 8: Change $1 - y$ to $1 - Y$.

Page 52, line 11: Change $<$ to $>$.

Page 54, line 7: Change $\overset{\text{cos}}{\text{cos}}$ to cos.

As to the supplements, we first note that Theorem 6.13 can be considerably extended, and should be changed to read as follows:

Theorem 6.13. *If* e *is aperiodic and positive, then*

$$\lim_{n \to \infty} P(V_n = k) = (EW_1)^{-1} q_k, \tag{6.29}$$

and thus

$$\lim_{n \to \infty} \frac{P(V_{n+m} = k)}{P(V_n = j)} = \frac{q_k}{q_j} . \tag{6.30}$$

More generally, we have that a necessary and sufficient

<u>condition for</u> (6.30) <u>to hold is that</u> (6.21) <u>be satisfied</u>.

<u>Proof</u>. We have

$$P(V_n = k) = P(\bar{Y}_{n+k-1} > k - 1) - P(\bar{Y}_{n+k} > k)$$

and as $P(\bar{Y}_{n+k} = r) = u_{n+k-r}\, q_r$, we have that

$$P(V_n = k) = \sum_{j=0}^{k} u_{n+k-j} - \sum_{j=0}^{k-1} u_{n+k-j-1}\, q_j. \tag{6.31}$$

Now if e is aperiodic and positive, (6.29) follows from (6.31) upon taking limits by the renewal theorem. If (6.21) is valid then (again from (6.31)) we have

$$\lim u_n^{-1}\, P(Y_{n+m} = k) = q_k,$$

which establishes (6.30) for the case of $j = 0$. The general case then follows at once from this result. Conversely if (6.30) is valid then

$$q_1 = \lim_{n\to\infty} \frac{P(V_n = 1)}{P(V_n = 0)} = \lim_n \left\{ \frac{u_{n+1}}{u_n}\, q_0 + q_1 - q_0 \right\},$$

and thus

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1.$$

As is clear from Sec. 6, a great many results depend on whether or not $\lim u_{n+1}/u_n = 1$. In general, this limit need not exist. In this regard, it might be of interest to point out that $\lim_n u_n^{1/n}$ always exists. More specifically, we have the following:

<u>Theorem.</u> <u>If</u> e <u>is an aperiodic recurrent event,</u> <u>then</u>

$$\lim_{n\to\infty} u_n^{1/n} = \frac{1}{R}$$

<u>where</u> R <u>is the radius of convergence of the series</u> $\sum_{n=0}^{\infty} u_n z^n$.

<u>Proof</u>. Let e be aperiodic and let $\limsup u_n^{1/n} = Y_R$. Then for infinitely many n we have for any ε, $0 < \varepsilon < 1/R$, that

$$u_n \geq (\tfrac{1}{R} - \varepsilon)^n.$$

Suppose the above is satisfied for $n = m$. Then for any $k = 0,1,2,\ldots,$ we have

$$u_{km} \geq (u_m)^k \geq (\tfrac{1}{R} - \varepsilon)^{km}.$$

But since e is aperiodic, we have $u_m > 0$ for $m \geq N_0$. If $n > 2m$ then $n = (k+1)m + r$, $k = 0,1,2,\ldots,$ $0 \leq r < m$, and thus

$$u_n \geq u_{km}u_{m+r} \geq (\tfrac{1}{R} - \varepsilon)^{km}\, u_{m+r} \geq (\tfrac{1}{R} - \varepsilon)^n\, A > 0,$$

where $A = \inf_{0 \leq r < m} u_{m+r} > 0$. Hence if $n \geq 2m > 2N_0$, we have

$$u_n^{1/n} \geq (\tfrac{1}{R} - \varepsilon),$$

and thus $\liminf u_n^{1/n} \geq 1/R - \varepsilon$. But as ε was arbitrary, we must have $\liminf u_n^{1/n} \geq 1/R$.

<u>Remark</u>. The above argument is a straight adaptation to the setting of general recurrent events of an argument due to Kakutani in [2] to prove the result for the special case of "return to zero," for integer-valued random variables.

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PREFACE

Part of the Project RAND program consists of basic supporting studies in mathematics. The mathematical research presented in this Memorandum deals with the theory of recurrent events, a subject of basic interest in the theory of probability and its military and scientific applications.

SUMMARY

The asymptotic behavior for large n of various quantities associated with a recurrent event is investigated. The results are applied to give information about certain functionals of sums of independent random variables.

CONTENTS

SOME LIMIT THEOREMS ASSOCIATED WITH A RECURRENT EVENT

1. INTRODUCTION

In this Memorandum we study the asymptotic behavior for large n of various random sequences associated with a recurrent event, extending results found in [4], [7], [8], [9], and [12]. The results found for general recurrent events are then applied to the study of various functionals of sums of independent random variables, with the explicit purpose of trying to encompass results on the fluctuation phenomena of sums within the framework of the general theory of recurrent events. However, these are to be looked on as examples only, since no attempt has been made to try to exhaust this method. Applications to other Markov processes are not considered, but no doubt could be given.

In summary, then, Sec. 2 is devoted to a review of the definitions and notational conventions which are used in this paper. In Sec. 3 criteria for transience, positivity, etc., are presented in terms of the sequence $\{EY_n\}$, where Y_n is the time, as observed from n, that the event last occurs. Section 4 is devoted to finding explicitly the joint limiting distribution of (N_n, Y_n), where Y_n is as above and N_n is the number of occurrences of the recurrent event by time n. In Sec. 5 other joint

limit distributions with N_n are given, and conditional limit distributions of N_n are found. Local limit theorems for the various random sequences associated with recurrent events are given in Sec. 6. In Sec. 7 some strong laws connected with N_n are investigated. The results of these sections are applied to sums of independent random variables in Sec. 8. The criteria of Sec. 3 are applied to give useful results on ladder points, which contain results implicit in the work of Spitzer in [17]. The joint limit distribution of the maximal partial sum and its time of occurrence, and the conditional limit distribution of the maximum given its time of occurrence, are given for the special case of summads of mean zero and finite variance. For integer valued S_n the recurrent event "$S_n = 0$," is also investigated. The paper concludes with an Appendix containing extensions of known Abelian and Tauberian theorems that are frequently useful in probability work and are needed in the proofs of some of the theorems in the paper.

2. PRELIMINARIES

Let e be a recurrent event on the nonnegative integers, with waiting times $\{W_k\}$, these being independent and identically distributed, positive, integer-valued, random variables

that can also assume the value ∞. We recall that e is called

(i) transient if $P(W_1 < \infty) = \rho < 1$,

(ii) certain if $\rho = 1$,

(iii) null if $\rho = 1$ and $EW_1 = \infty$,

(iv) positive if $\rho = 1$ and $EW_1 < \infty$,

(v) periodic of period r if e can only occur at times nr for $n = 0, 1, 2, \ldots$.

Definitions.

(2.1) $N_n = \sup \{k: W_1 + \ldots + W_k \leq n\}$ (number of occurrences by time n),

(2.2) $Y_n = W_1 + W_2 + \ldots + W_{N_n}$ (time of last occurrence),

(2.3) $V_n = W_1 + \ldots + W_{N_n+1} - n$ (time till next occurrence as measured from n).

For $|t| < 1$, denote

$$(2.4)\quad F(t) = E[t^{W_1}; W_1 < \infty] = \sum_{k=1}^{\infty} P(W_1 = k)t^k,$$

$$(2.5)\quad U(t) = \sum_{n=0}^{\infty} P(Y_n = n)t^n \quad (Y_0 = 0),$$

$$(2.6)\quad T(t) = \sum_{n=0}^{\infty} P(Y_n = 0)t^n = \sum_{n=0}^{\infty} P(W_1 > n)t^n.$$

In addition, let

$$u_n = P(Y_n = n), \quad q_n = P(Y_n = 0).$$

Unless otherwise specified, t will always be such that $|t| < 1$ and x, y will be quantities such that $|x| \leq 1$ and $|y| \leq 1$. If X is a random variable and A an event we shall denote $\int_A X \, dP$ by $E(X; A)$.

3. CRITERIA FOR RECURRENT EVENTS

Lemma 3.1.

$$(3.1) \qquad \sum_{n=0}^{\infty} t^n E(y^{N_n} x^{Y_n}) = T(t)[1 - yF(xt)]^{-1} .$$

Proof. $P(N_n = k, Y_n = r) = P(W_1 + \dots + W_k = r)P(W_{k+1} > n - r)$, and 3.1 follows upon taking generating functions. We shall need (3.1) in the next section. All we need here is the special case with $y = 1$, which is

$$(3.2) \qquad \sum_{n=0}^{\infty} t^n E\, x^{Y_n} = T(t)[1 - F(xt)]^{-1} .$$

Theorem 3.1. Let $\Delta_k = E(Y_k - Y_{k-1})$. Then

$$(3.3) \qquad U(t) = \exp \sum_{k=1}^{\infty} t^k/k \; \Delta_k .$$

Remark. This curious exponential identity shows that knowledge of $\{E\,Y_n\}$ for all n completely determines the recurrent event e. It came to our attention through Professor M. Dwass.

Proof. Differentiate (3.2) with respect to x at 1 to obtain

$$(3.4) \qquad (1 - t) \sum_{n=0}^{\infty} t^{n-1}\, E\, Y_n = U'(t)/U(t).$$

Thinking of the left hand side as a known function of t, we see that (3.4) is a differential equation for the function U(t). Since U(0) = 1, the right-hand side of (3.3) is the unique solution to this equation. We are now in a position to prove the following result.

Theorem 3.2. Suppose that e is a recurrent event and that $\Delta_k = E(Y_k - Y_{k-1})$. $(\Delta_0 = 0)$. Then

$$(3.5) \qquad 1 - \rho = P(W = \infty) = e^{-\sum_{k=1}^{\infty} \Delta_k/k}$$

If e is aperiodic, then

$$(3.6) \qquad EW_1 = e^{\sum_{k=1}^{\infty} (1 - \Delta_k)/k},$$

<u>in the sense that both sides are finite or infinite and always equal</u>.

<u>Proof</u>. Since $\Delta_k \geq 0$, (3.5) follows from (3.3) and a generalized version of Abel's theorem. Now suppose e is aperiodic and that $\Sigma(1 - \Delta_k)/k$ converges. Then

$$EW_1 = \lim_{t\to 1^-} T(t)$$

$$= \lim_{t\to 1^-} e^{\Sigma t^k/k}\, e^{-\Sigma\, \Delta_k\, t^k/k}$$

$$= \lim_{t\to 1^-} e^{\sum_{k=1}^{\infty} \frac{1 - \Delta_k}{k} t^k} .$$

Since e^x is continuous in x, the above limit is $e^{\sum_{k=1}^{\infty} (1-\Delta_k)/k}$ by Abel's theorem. To establish the converse, we must show that

$$\lim_{t\to 1} \sum_{k=1}^{\infty} \frac{1 - \Delta_k}{k} t^k = EW_1 < \infty$$

implies that $\Sigma \frac{1 - \Delta_k}{k}$ converges. If we could show that $\lim_{k\to\infty} \Delta_k = 1$, then the desired conclusion would follow at once from Tauber's theorem. To see that $\lim_{k\to\infty} \Delta_k = 1$, observe that

$$(3.7) \qquad (1 - t) \Sigma \Delta_n t^n = tF'(t)U(t)(1 - t) ,$$

and

$$F'(1) = EW_1 < \infty ,$$

and that the series $U(t)(1 - t)$ becomes at 1 the series

$$u_1 + (u_2 - u_1) + \ldots = \lim_{n\to\infty} u_n ,$$

where $u_n = P(Y_n = n)$. Now, by a well-known renewal theorem (see [8]) we have $\lim_{n\to\infty} u_n = 1/EW_1$, and so the series in (3.7) converges at 1 to 1 by Merten's theorem. This completes the proof.

As a corollary of the proof we have the following.

Corollary 3.3. *If* $EW_1 < \infty$ *and* e *is aperiodic, then*

$$\lim_{n\to\infty} \Delta_n = \lim_{n\to\infty} E(Y_n - Y_{n-1}) = 1,$$

and the series $\sum_{n=1}^{\infty} \frac{(1 - \Delta_n)}{n}$ *converges to a finite positive value*.

Theorem 3.4. *Suppose* e *is a null recurrent event. If for some* β $(0 \leq \beta \leq 1)$ *we have*

$$(3.8) \qquad \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \Delta_k = \lim_{n\to\infty} Y_n/n = \beta ,$$

then

$$1 - F(t) = (1-t)^{\beta} L[1/(1-t)] ,$$

where

$$(3.9) \qquad L[1/(1-t)] = e^{\sum_{k=1}^{\infty} (\beta - \Delta_k) t^k/k}$$

is a slowly varying function of $1/(1-t)$.* *Conversely, if*

$$1 - F(t) = (1-t)^{\beta} h[1/(1-t)]$$

for some β $(0 \leq \beta \leq 1)$ *and slowly varying function* h, *then* (3.8) *holds and we must then have* $h(x) = L(x)$, *where* $L(x)$ *is given in* (3.9).

Proof. Lamperti in [10] showed that (3.8) and

$$1 - F(t) = (1 - t)^{\beta} h\left(\frac{1}{1-t}\right)$$

were equivalent, where $h(x)$ is a slowly varying function. To identify $h[1/(1-t)]$ as $L[1/(1-t)]$, observe that by (3.3), we have

$$(1 - t)^{-\beta}[1 - F(t)] = e^{-\beta \ln(1-t)} e^{-\Sigma t^k/k \, \Delta_k}$$

$$= e^{\beta t^k/k} e^{-\Sigma t^k/k \, \Delta_k} ,$$

which establishes the result.

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*A function $L(x)$ is called slowly varying if it is positive for x sufficiently large, and if for each positive a, we have $L(ax)/L(x) \to 1$ as $x \to \infty$.

<u>Remark</u>. When $EW_1 < \infty$, $\beta = 1$ and we have shown that $\lim_{k\to\infty} \Delta_k = 1$. It is an open question as to when this is true for $0 \leq \beta < 1$. (See, however, theorem 8.3.) In line with this remark it might be well to point out that there not only exist recurrent events for which $\lim_{k\to\infty} \Delta_k = \beta$, but there is even a recurrent event for which $\Delta_k \equiv \beta$. To see this, observe that if $0 < \beta < 1$, then if $\Delta_k \equiv \beta$ we have by (3.10) that

$$F(t) = 1 - (1 - t)^{\beta} .$$

Thus $f_k = (-)^k \binom{\beta}{k}$, which is positive and < 1, and $F(1) = \sum_{k=1}^{\infty} f_k = 1$, showing that these $\{\Delta_k\}$ do indeed determine a recurrent event.

4. THE JOINT DISTRIBUTION OF (N_n, Y_n).

This first theorem is only to establish the case of greatest interest.

<u>Theorem 4.1</u>. <u>If</u> e <u>is transient, then</u> (N_n, Y_n) <u>converges with probability one to a finite random vector</u> (N, Y), <u>and</u>

$$(4.1) \qquad E(x^Y y^N) = (1 - \rho)[1 - yF(x)]^{-1} .$$

If e *is recurrent, then* (N_n, Y_n) *converge to* ∞ *with probability one, and if* e *is positive, then*

$$(4.2) \qquad (N_n/n,\ Y_n/n) \longrightarrow (1/EW_1,\ 1)$$

with probability one.

Proof. Since a transient event takes place finitely often with probability one, (4.1) follows from (3.3); (4.2) is well known (see e.g., [6]).

Thus, the case of greatest interest will be the null-recurrent case. The limit marginal distributions for this case were first found by Feller in [9] for N_n and independently by Lamperti in [12] and Dynkin in [8] for Y_n.

Theorem 4.2. *Suppose* e *is a null-recurrent event. Then in order that there exist constants* $a_n \geq 0$, $b_n \geq 0$ *such that* $(N_n/b_n, Y_n/a_n)$ *should converge in distribution to* (N, Y) *having a nondegenerate distribution, it is necessary and sufficient that*

$$(4.3) \qquad 1 - F(t) = (1 - t)^{\beta} L\left(\frac{1}{1-t}\right)$$

for some β, $0 < \beta < 1$, *and some slowly varying function* $L(x)$. *If condition* (4.3) *is satisfied, then we may choose*

$$(4.4) \qquad a_n = n, \quad b_n = n^{\beta}/L(n)$$

and the distribution of (N, Y) *will be uniquely determined by its moments*:

$$(4.5)\qquad E(N^m Y^k) = \frac{(-)^k\, k\,!\, m\,!}{\Gamma(m\beta+k+1)} \binom{-\beta(m+1)}{k} .$$

(Remark: An explicit formula for the (N, Y) distribution will be given below in Corollary 4.3).

Proof. Necessity: This follows at once from known facts. If $(N_n/b_n, Y_n/a_n)$ is to have a nondegenerate limit distribution, then clearly N_n/b_n must have one. But it was shown in [4] that this is true if and only if condition (4.3) holds for some β, $0 < \beta \leq 1$.

Sufficiency: By (3.3) we have

$$(4.6)\qquad \sum_{n=0}^{\infty} E[x^{Y_n} y^{N_n}] = T(t)[1 - yF(xt)]^{-1} ,$$

which is an analytic function in (x, y, t) for $|x| \leq 1$, $|y| \leq 1$, $|t| < 1$. Taking the m-th derivative of (4.6) with respect to y at 1 results in

$$(4.7)\qquad \sum_{n=0}^{\infty} t^n E(x^{Y_n} N_n^{(m)}) = m!F(tx)^m[1 - F(t)](1 - t)^{-1} \cdot [1 - F(xt)]^{-(m+1)} ,$$

where

$$N_n^{(m)} = N_n (N_n - 1) \dots (N_n - m+1) .$$

Set $x = e^{-\lambda(1-t)}$ in (4.7) and expand in powers of λ to obtain

$$(4.8) \quad \sum_k (-)^k/k! \; \lambda^k \, (1-t)^k \sum_{n=0}^{\infty} t^n \, E(Y_n^k \, N_n^{(m)})$$

$$= [1 - F(t)](1-t)^{-1} F(te^{-\lambda(1-t)})^m [1 - F(te^{-\lambda(1-t)})]^{-(m+1)} .$$

As $t \to 1^-$,

$$1 - e^{-\lambda(1-t)} \sim (1 + \lambda)(1 - t) ,$$

and so, taking account of the slowly varying nature of L, we obtain from (4.8) and (4.3) (after a slight rearrangement)

$$(4.9) \quad \lim_{t \to 1} \Sigma \, \lambda^k (1 - t)^{k+1} (1 - t)^{\beta m} \, L\left(\frac{1}{1-t}\right)^m \sum_{n=0}^{\infty} t^n \, E(Y_n^k \, N_n^{(m)})$$

$$= (-)^k \, k! m! \, (1 + \lambda)^{-\beta(m+1)} .$$

Since the quantity in (4.8) is analytic in λ, t for $|t| < 1$ and $\lambda \geq 0$, and the right-hand side of (4.9) is analytic in λ at $\lambda = 0$, we obtain from (4.9)

$$(4.10)\quad \lim_{t\to 1^-} (1-t)^{k+\beta m} L\left(\frac{1}{1-t}\right)^m \sum_{n=0}^{\infty} t^n E(Y_n^k N_n^{(m)} - Y_{n-1}^k N_{n-1}^{(m)})$$

$$= (-)^k k!\, m! \binom{-\beta(m+1)}{k} .$$

Now, $E(Y_n^k N_n^{(m)})$ is, for each fixed k and m, a monotone increasing function of n. Hence we may apply Karamata's Tauberian theorem (see [5], p. 507) to (4.10) and conclude that

$$(4.11)\quad E(N_n^{(m)} Y_n^{(k)}) \sim n^{\beta m} n^k L(n)^{-m} \frac{(-)^k k!\, m!}{\Gamma(m\beta+k+1)} \binom{-\beta(m+1)}{k} .$$

Finally, as $E(N_n^{(m)} Y_n^k) \sim E(N_n^m Y_n^k)$, $n \to \infty$, we have (4.5). To complete the proof, we must show that these moments uniquely determine a distribution. By theorem 1.12 of [15] this will be true provided that the series

$$(4.12)\quad \sum_{n=0}^{\infty} E(N^{2n} + Y^{2n})^{\ 1/2n} = \infty .$$

A simple computation shows that

$$E(N^{2n} + Y^{2n}) \sim 1/2, \quad n\to\infty$$

and so (4.12) certainly holds. This completes the proof.

If we set $k = 0$ in (4.5), we obtain the known result that

$$(4.13) \qquad EN^m = 1/\Gamma(\beta m+1) \ .$$

These are known to be the moments of a distribution on the positive axis with density

$$(4.14) \quad g_\beta(x) = (\pi\beta)^{-1} \sum_{j=1}^{\infty} (-)^{j-1}/j! \ \Gamma(\beta j+1) \sin \pi\beta j \ x^{j-1} \ .$$

(See [4]). If we set $m = 0$ in (4.5), we obtain the known result that

$$(4.15) \qquad EY^k = (-)^k \binom{-\beta}{k} ,$$

which are seen to be the moments of a distribution on the interval [0, 1] with a density

$$(4.16) \qquad f_\beta(x) = \sin \pi\beta/\pi \ x^{\beta-1} \ (1-x)^{-\beta} \ .$$

It is possible to give an interesting characterization of the distribution of (N, Y) and even to write down its density function.

Corollary 4.2.* *(N, Y) is distributed like $(\tilde{N}\, Y^{\beta}, Y)$, where $(\tilde{N}, Y)$ are independent, and Y has density given in (4.16), while $\tilde{N}$ is a positive random variable with a distribution uniquely determined by its moments:*

$$(4.17) \qquad E\tilde{N}^m = \frac{(m+1)!\ \Gamma(\beta+1)}{\Gamma(1+\beta(m+1))} \ .$$

These are the moments of a distribution on the positive axis with density

$$(4.18) \qquad h_\beta(x) = \Gamma(\beta+1)\ x\ g_\beta(x),$$

where $g_\beta(x)$ is given in (4.14).

Proof. A simple calculation shows that

$$(4.19) \quad \frac{(-)^k\ k!\ m!}{\Gamma(m\beta+k+1)} \binom{-\beta(m+1)}{k} = \frac{(m+1)!\ \Gamma(\beta+1)}{\Gamma(1+\beta(m+1))}\ \frac{\Gamma(\beta(m+1))}{\Gamma(\beta)\Gamma(\beta m+k+1)}$$

and

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*We are indebted to Prof. M. Dwass for bringing this corollary to our attention.

$$(4.20) \qquad EY^{\beta m+k} = \frac{\Gamma(k+\beta(m\Sigma 1))}{\Gamma(\beta)\Gamma(m\beta+k+1)} .$$

The result now follows from (4.20), (4.19), and (4.17). Using this corollary, we have (N, Y) has a density on $(0, \infty) \times [0, 1]$ which is

$$(4.21) \quad K_\beta(x, y) = h_\beta(xy^{-\beta})\, y^{-\beta}\, f_\beta(y) =$$

$$\Gamma(\beta) \sin \pi\beta/\pi^2 \sum_{j=1}^{\infty} (-)\ j\text{-}1/j!\ \sin \pi\beta j\Gamma(\beta j+1)x^{-j}y^{-\beta j-1}(1-y)^{-\beta}.$$

The case of $\beta = 1/2$ is of special interest. For this case we have

$$(4.22) \qquad \text{(i)} \quad g_{1/2}(x) = \pi^{-1/2}\, e^{-\,x^2/4}$$

$$\text{(ii)} \int_0^x h_{1/2}\,(x)dx = 1 - e^{-\,x^2/4}$$

$$\text{(iii)}\ K_{1/2}(x, y) = x/2\pi\ e^{-\,x^2/4y}\ y^{-1}\ (1-y)^{-1/2} .$$

<u>Corollary 4.3.</u> <u>Let</u> $\bar{Y}_n = n - Y_n$ <u>(time since last occurrence of</u> e). <u>Then condition</u> (4.3) <u>is both necessary and sufficient for</u> $(N_n/b_n,\ \bar{Y}_n/n)$ <u>to converge in distribution to</u> $(N, \bar{Y})$ <u>having a nondegenerate distribution. Moreover we have</u>

$$(4.23) \qquad E(N^m \bar{Y}^k) = \frac{m!\ k!\ (-)^k \binom{\beta-1}{k}}{\Gamma(k+1+\beta(m+1))}$$

<u>which are the moments of a distribution on</u> $[0, \infty) \times [0, 1]$ <u>with density</u>

$$(4.24) \qquad \bar{K}_{1-\beta}(x, y) = K_\beta(x, 1-y)$$

<u>where</u> $K_\beta(x, y)$ <u>is given in Eq</u>. (4.21).

<u>Proof</u>. We have

$$(4.25) \quad \lim_{n\to\infty} E\left[(N_n/b_n)^m (\bar{Y}_n/n)^k\right] = \lim_{n\to\infty} E\left[(N_n/b_n)^m \left(1 - Y_n/n\right)^k\right]$$

$$= E\left(N^m (1-y)^k\right) = E\tilde{N}^m\ E\ Y^{\beta m}\ (1-y)^k\ .$$

A simple computation now completes the proof.

5. OTHER GLOBAL LIMIT DISTRIBUTIONS

From the joint distribution of (N, Y) we may obtain the joint limit distribution of $(N_n/b_n, V_n/n)$ and

$$(N_n/b_n, \frac{N_{n+[rn]} - N_n}{b_n}\ .$$

Since the proofs of these follow

closely those used by Lamperti in [12] for V_n/n and Dynkin in [8] for $\frac{N_{n+[rn]} - N_n}{b_n}$, respectively, no proofs will be included.

<u>Theorem 5.1</u>. <u>Condition</u> (4.3) <u>is both necessary and sufficient for</u> $(N_n/b_n, V_n/n)$ <u>to have a nondegenerate limit distribution</u>. <u>The density of the limit distribution is</u>

$$(5.1) \qquad h_\beta\,(x(1+u))(1+u)^\beta\, f_\beta(1/1+u)(1+u)^{-2}\ ,$$

<u>where</u> h_β <u>and</u> f_β <u>are given in</u> (4.8) <u>and</u> (4.16, <u>respectively</u>.

<u>Theorem 5.2.</u> <u>Under condition</u> (4.3),

$$(5.2) \quad \lim_{n\to\infty} P\left(N_n/b_n \le x,\ \frac{N_{n+[rn]} - N_n}{b_n} > u\right)$$

$$= \int_0^x \int_0^r G_\beta\left(\frac{r-s}{u^{1/\beta}}\right) K_\beta(y,\ s)\ dy\ ds\ ,$$

<u>where</u> K_β <u>is given in</u> (4.21) <u>and</u> $G_\beta(x)$ <u>is the stable law of index</u> β <u>with Laplace transform</u>

$$\int_0^\infty e^{-\lambda x}\, dG_\beta(x) = e^{-\lambda^\beta}\ .$$

From corollary 4.2, it follows at once that

$$P(N \leq x \mid Y = y) = P(\tilde{N} \leq xy^{-B}) ,$$

and so $P(N \leq x \mid Y = y)$ is a continuous function of (x, y), $0 \leq x < \infty$, $0 < y \leq 1$. This makes it plausible that for b_n given in (4.4), we have

$$(5.3) \quad \lim_{n\to\infty} P(N_n/b_n \leq x \mid Y_n = [sn]) = P(\tilde{N} \leq x\, s^{-\beta})$$

when condition (4.3) holds. Let us first observe that if (5.3) is to hold for all s, $0 < s \leq 1$, then condition (4.3) is necessary. However, we have been unable to show that (4.3) is sufficient, and can only establish (5.3) under a more stringent condition.

Theorem 5.3. *Suppose* e *is a null-recurrent event such that*

$$(5.4) \qquad P(Y_n = n) \sim n^{\beta-1}/\Gamma(\beta)\ L(n)^{-1}, \quad n\to\infty ,$$

where $L(n)$ *is a slowly varying function.** *Then* (5.3) *holds for each* s, $0 < s \leq 1$, *and in fact uniformly for* $0 < \epsilon \leq s \leq 1$.

Proof. $P(N_n/b_n \leq x \mid Y_n = [sn]) = P(N_{[sn]}/b_n \leq x \mid Y_{[sn]} = [sn])$ and so

$$\lim_{n\to\infty} P(N_n/b_n \leq x \mid Y_n = [sn]) = \lim_{n\to\infty} P(N_n/b_{[n/s]} \leq x \mid Y_n = n) .$$

*By Theorem 3.4, $L(x)$ must then be given by Eq. (3.9).

Now, by known properties of slowly varying functions (See [11]), we have $b_{[n/s]}b_n^{-1} \to s^{-\beta}$ for $0 < s \leq 1$, and uniformly so for $0 < \epsilon \leq s \leq 1$. Thus all assertions of Theorem 5.3 will be established if we show that under (5.4),

$$(5.5) \quad \lim_{n\to\infty} P(N_n/b_n \leq x \mid Y_n = n) = P(\tilde{N} \leq x) \ .$$

In [7], Dwass has shown this to be the case if $L(n)$ is a constant. To establish (5.5) for a general $L(n)$ we need the following:

Lemma 5.4. If C_n is the $(m+1)$-st convolution of the sequences $\{P(Y_n = n\}$ with itself, and if condition (5.4) holds, then

$$C_n \sim \frac{n^{\beta m-1} L(n)^{-(m+1)}}{\Gamma(\beta)\Gamma(\beta(m+1))} \ .$$

Proof. The proof is a direct consequence of Corollary (A.5) in the appendix.

To complete the proof of (5.5), we observe that

$$\Sigma\, t^n E(x^{N_n}; Y_n = n) = [1 - xF(t)]^{-1} \ ,$$

and thus

$$\Sigma\, t^n\, E[N_n^{(m)};\ Y_n = n] = m!\ [1 - F(t)]^{-(m+1)},$$

from which it follows that

$$E(N_n^{(m)};\ Y_n = n) = m!\ C_n,$$

where C_n is given in the lemma above. Hence,

$$E(N_n^{(m)} | Y_n = n) \sim \frac{\Gamma(\beta)m!}{\Gamma(\beta(m+1))}\ [n^\beta/L(n)]^m$$

$$= \frac{(m+1)!\Gamma(1+\beta)}{\Gamma(1+\beta(m+1))}\ [n^\beta/L(n)]^m\ ,$$

which by (4.17) are the moments of $\tilde{N}$.

Remark. At this point we shall pause to point out the connection between condition (5.4) and condition (4.3). If (5.4) holds for $0 < \beta \leq 1$, then a well known Abelian theorem (see [5], p. 460) assures us that

$$(5.6) \qquad 1 - F(t) \sim (1-t)^\beta\ L(1/1-t),\quad t \to 1^-$$

and thus condition (5.4) holds. On the other hand, if (4.3) holds, then in general one may conclude (via Karamata's Tauberian theorem) only that

$$(5.7) \qquad \sum_{k=0}^{n} P(Y_k = k) \sim \frac{n^{\beta} L^{-1}(n)}{\Gamma(1+\beta)}, \qquad n\to\infty .$$

On the other hand if we know that $P(Y_k = k)$ is a monotone function in k, then we may conclude (see Theorem A.3 in the Appendix) that

$$(5.8) \qquad P(Y_n = n) \sim \frac{n^{\beta-1} L^{-1}(n)}{\Gamma(\beta)} .$$

For future reference let us record at this point the following equivalence relation: (5.6) and

$$(5.9) \qquad P(Y_n = 0) \sim \frac{n^{-\beta} L(n)}{\Gamma(1-\beta)}$$

are equivalent. To see this, observe that if (5.6) holds, then

$$T(t) \sim (1-t)^{\beta-1} L(\tfrac{1}{1-t})$$

and thus by Karamata's theorem

$$P(Y_1 = 0) + \dots + P(Y_n = 0) \sim \frac{n^{1-\beta} L(n)}{\Gamma(1-\beta)} .$$

But, since $P(Y_n = 0)$ is monotone, we have by Corollary A.3 that (5.9) holds. Conversely, if (5.9) holds, then the

Abelian theorem mentioned above shows that

$$T(t) \sim (1-t)^{\beta-1} L\left(\frac{1}{1-t}\right)$$

and thus (5.6) holds.

Theorem 5.5. *Suppose condition* (5.4) *holds. Then for* $0 \leq \alpha \leq 1$, $0 < s \leq \alpha$, *and* $b_n = n^{\beta}/L(n)$, *we have*

$$\lim_{n\to\infty} P(N_n/b_n \leq x \mid Y_{[n\alpha]} = [sn]) = P(N_{\alpha,s} \leq x), \tag{5.10}$$

where $N_{\alpha,s}$ *is distributed like*

$$s^{\beta} \tilde{N}_1 + (1-\alpha)^{\beta} N_1 .$$

Here $\tilde{N}_1$ *and* N_1 *denote two random variables independent of each other and distributed like* $\tilde{N}$ *and* N *respectively.*

Proof. For $\lambda \geq 0$ we have

$$E\left(e^{-\lambda N_n/b_n} \mid Y_{[n\alpha]} = [ns]\right) \tag{5.11}$$

$$= E\left[e^{-\lambda N_{[sn]}/b_n} \mid Y_{[ns]} = [ns]\right] E\left[e^{-\lambda N_{n-[\alpha n]}/b_n}\right].$$

By the remark following Theorem 5.4 we have that condition (4.3) holds. Hence we may apply Theorems 5.3 and 4.2 to Eq. (5.11) to conclude that

$$\lim_{n\to\infty} E[e^{-\lambda N_n/b_n} \mid Y_{[n\alpha]} = [ns]] = E[e^{-\lambda s^\beta \tilde{N}}]E(e^{-\lambda(1-\alpha)^\beta N}),$$

which establishes the assertion made in (5.10).

6. LOCAL LIMIT THEOREMS

In this section we shall investigate various "local limit theorems" for the functionals associated with a recurrent event. We start our investigation with N_n, where we have the following general result.

Theorem 6.1. *For any recurrent event*

$$\lim_{n\to\infty} \frac{\sum_{r=0}^{n+m} P(N_r = k)}{\sum_{r=0}^{n} P(N_r = j)} \quad \rho^k/\rho^j$$

where m *is fixed and* ρ *is given in* Eq. 3.5.

Proof. It is clearly sufficient to establish (6.1) for the special case of $j = 0$. From the relation

$$P(N_n = k) = \sum_{j=0}^{n} P(W_1 + \dots + W_k = n-j)\, q_j ,$$

we obtain

$$\sum_{r=0}^{n} P(N_r = k) = \sum_{j=0}^{n} P(W_1 + \dots + W_k \le n-j)\, q_j .$$

But for each fixed r we have

$$q_{n+r}\Big(\sum_{j=0}^{n} P(W_1 + \dots + W_k \le n-j)\Big)^{-1} \to 0 \text{ as } n\to\infty ,$$

and thus by an elementary theorem on Norlünd summability (see [3], p. 20) we have

$$\lim_{n\to\infty} \frac{\sum_{r=0}^{n+m} P(N_r = k)}{\sum_{r=0}^{n} P(N_r = 0)} = \lim_{n\to\infty} \frac{\sum_{j=0}^{n+m} P(W_1 + \dots + W_k \le n-j)\, q_j}{\sum_{j=0}^{n} q_j}$$

$$= \lim_{n\to\infty} P(W_1 + \dots + W_k \le n) = \rho^k ,$$

which establishes (6.1) for the case of $j = 0$.

Under certain general conditions it is possible to greatly improve upon Theorem 6.1. Thus we have the following:

Theorem 6.2. If e is transient, then

$$\lim_{n\to\infty} P(N_n = k) = \rho^k(1-\rho)$$

and thus

$$\lim_{n\to\infty} \frac{P(N_{n+m} = k)}{P(N_n = j)} = \rho^k/\rho^j . \tag{6.2}$$

If e is null and

$$(6.3) \qquad 1 - F(t) \sim (1-t)^{\beta}\, L\left(\frac{1}{1-t}\right),$$

where $0 \leq \beta < 1$ and $L(x)$ is a slowly varying function,* then

$$(6.4) \qquad \lim_{n\to\infty} \Gamma(1-\beta) n^{\beta} L(n)^{-1}\, P(N_n = k) = 1,$$

and thus (6.2) holds in this case as well.

Proof. If e is transient, then it follows from Theorem 4.1 that N_n converges to a finite random variable with a geometric distribution of parameter ρ. Now suppose (6.3) is satisfied. If we set $x = 1$ in Eq. (3.1), we obtain

$$\sum_{n=0}^{\infty} t^n\, E\, y^{N_n} = [1-F(t)](1-t)^{-1}[1-yF(t)]^{-1}$$

$$\sim (1-t)^{\beta-1}\,(1-y)^{-1}\, L\left(\frac{1}{1-t}\right), \quad t \to 1^- .$$

Thus by Karamata's Theorem we have

$$\sum_{r=0}^{n} E\, y^{N_r} \sim \frac{n^{1-\beta}\, L(n)\, (1-y)^{-1}}{\Gamma(2-\beta)} .$$

But, for $0 < y < 1$ we have that $E\, y^{N_n}$ is monotone in n, and

*L(x) must then be given by Eq. (3.9).

thus we may apply Corollary A.3 in the Appendix to conclude that

$$(6.5)\qquad \lim_{n\to\infty} \Gamma(1-\beta)n^{\beta} L(n)^{-1} E\, y^{N_n} = (1-y)^{-1} .$$

Let

$$a_{n,k} = \Gamma(1-\beta)n^{\beta} L(n)^{-1} P(N_n = k) .$$

Now, it follows from (6.5) that $a_{n,k}$ is bounded for each value of k, and thus we may extract a subsequence (by the diagonal procedure) $a_{n_r,k}$ which converges to, say, a_k as $r\to\infty$. But then it follows (again from (6.5)) that $a_k = 1$. If then there were another convergent subsequence of $a_{n,k}$, the same argument as used above would show its limit to be 1. Thus we have

$$\lim_{n\to\infty} a_{n,k} = 1,$$

which is the same as the assertion made in (6.4).

When condition (6.3) holds (or more generally when (6.2) holds in the recurrent case) we have the following curious result.

<u>Corollary 6.3</u>. <u>If</u> e <u>is certain and</u> (6.2) <u>holds, then</u> <u>for</u> $0 \le j \le k$ <u>we have</u>

$$(6.6) \qquad \lim_{n\to\infty} P(N_n = j \mid N_n \leq k) = (k+1)^{-1} .$$

Proof. If e is recurrent, then $\rho = 1$, and if (6.2) holds, we have

$$\lim_{n\to\infty} \sum_{r=0}^{k} P(N_n = r)/P(N_n = j) = k+1.$$

From this (6.6) is evident.

Remark. It is easy to exhibit aperiodic positive events for which (6.2) fails to hold, and it can be shown that (6.2) may also fail for null events.*

Turning our attention next to Y_n, we have the following result.

Theorem 6.4. *For any recurrent event,*

$$(6.7) \qquad \lim_{n\to\infty} \frac{\sum_{r=0}^{n+m} P(Y_r = k)}{\sum_{r=0}^{n} P(Y_r = j)} = u_k/u_j .$$

Proof. (6.7) is a simple consequence of the following two facts:

$$(6.8) \qquad P(Y_n = k) = u_k \, q_{n-k}$$

*The null event given in the Note proceeding Theorem 6.5 provides an example.

and

$$(6.9)\qquad \lim_{n\to\infty} \left(\sum_{r=0}^{n+m} q_{r-k}\right)\left(\sum_{r=0}^{n} q_r\right)^{-1} = 1.$$

Again, as in the case of N_n, we may improve upon this result in many instances. Thus, if e is transient, we have $P(Y_n = k) \to u_k(1-\rho)$ as $n\to\infty$ and thus

$$(6.10)\qquad \lim_{n\to\infty} \frac{P(Y_{n+m} = k)}{P(Y_n = j)} = u_k/u_j \ .$$

More generally, it is easily seen from (6.8) that in order that (6.10) hold, it is both necessary and sufficient that

$$(6.11)\qquad \lim_{n\to\infty} \frac{q_{n+1}}{q_n} = 1 \ .$$

In the transient case this is always true and in the certain case we have the following simple sufficient conditions.

Theorem 6.5. If e is certain, then (6.11) holds if either of the following two conditions is satisfied;

(i) $q_n \sim n^{-\alpha} L(n)$ for $\alpha > 0$ and some slowly varying function $L(x)$.

(ii) e is null and u_n is monotone.

Note. If $0 \le \alpha < 1$, then by the remark following Theorem 5.5 we have that (i) and condition (4.3) are equivalent, and in this case $L(n)$ is given by (3.9).

It is easy to see that (6.11) may fail in the positive aperiodic case. Indeed, it may not even be possible to write the ratio in (6.11) (e.g., if W_1 is bounded). In the null case the following example shows that (6.11) may also fail. Take $P(W_1 = k) = 0$ unless $k = 2^j$ and $P(W_1 = 2^j) = (1/2)^j$. Then $q_n = (1/2)^r$ if $2^r \le n < 2^{r+1}$. We then have $q_n/q_{n-1} = 1$ i.o. and thus $\limsup q_n/q_{n-1} \ge 1$. On the other hand $q_n/q_{n-1} = 1/2$ if $n = 2^r$ for $r = 0, 1, 2, \ldots$, and thus $\liminf q_n/q_{n-1} \le 1/2$.

Proof. If (i) holds, it is easy to see that (6.11) is true; so we need only establish (6.11) when (ii) holds. If e is null, then $q_n > 0$ for all n. For if not, then for some n_o we have $q_{n_o} = 0$ and thus $q_n = 0$ for all $n \ge n_o$ which would give $P(W_1 \le n_o) = 1$. Consequently $EW_1 < \infty$, a contradiction. From the monotoneity of q_n we have

$$(6.12) \qquad q_{n+1}/q_n \le 1 .$$

From the relation

$$1 = \sum_{k=0}^{n} u_{n-k}\, q_k ,$$

it easily follows that

$$q_n = \sum_{k=0}^{n-1} q_k(u_{n-1-k} - u_{n-k}) \geq q_{n-1}(1-u_n).$$

By the renewal theorem (see [3], p. 26) we have $u_n \to 0$, and thus

(6.13) $$\liminf q_n/q_{n-1} \geq 1.$$

Combining (6.12) and (6.13), we obtain (6.11).

Corollary 6.6. *If* (6.11) *holds, then for* $0 \leq j \leq k$ *we have*

(6.14) $$\lim_{n\to\infty} P(Y_n = j \mid Y_n \leq k) = u_j(u_0 + \dots + u_k)^{-1} .$$

Proof. Obvious.

If e is null and satisfies condition (5.4), then we have the following local version of the generalized arc sin laws for Y_n.

Theorem 6.7. Assume condition (5.4) *holds. Then*

$$(6.15) \qquad P(Y_n = k)k^{1-\beta}(n-k)^{\beta} = \frac{\sin \pi \beta}{\pi} + o(1)$$

where $o(1)$ *converges to zero uniformly in* n, k *as* $\min (k, n-k) \to \infty$.

Proof. By the remark following Theorem 5.5 we have that if (5.4) holds, then

$$q_n \sim \frac{n^{\beta} L(n)}{\Gamma(1-\beta)} .$$

Thus, from (6.14) we have

$$(6.16) \quad P(Y_n = k) \sim k^{\beta-1}(n-k)^{-\beta} L(k)^{-1} L(n-k) \frac{1}{\Gamma(\beta)\Gamma(1-\beta)}$$

as $\min (k, n-k) \to \infty$. But, since $L(k)^{-1} L(n-k) \to 1$ uniformly as $\min (k, n-k) \to \infty$ and

$$\frac{1}{\Gamma(\beta)\Gamma(1-\beta)} = \frac{\sin \pi \beta}{\pi} ,$$

we see that (6.15) follows from (6.16).

<u>Corollary 6.8</u>. <u>If</u> $k \sim [sn]$ <u>for</u> $0 < s \leq 1$, <u>then</u>

$$\lim_{n \to \infty} nP(Y_n = k) = \frac{\sin \pi \beta}{\pi} s^{\beta-1} (1-s)^{-\beta} \tag{6.17}$$

<u>and the convergence is uniform in</u> s <u>for</u> $0 \leq \epsilon \leq s \leq 1$.

<u>Proof</u>. A direct consequence of Theorem 6.7.

Turning our attention to $\bar{Y}_n = n - Y_n$, we have the following result.

<u>Theorem 6.9</u>. <u>For any recurrent event</u> e

$$\lim_{n \to \infty} \frac{\sum_{r=0}^{n+m} P(\bar{Y}_r = k)}{\sum_{r=0}^{n} P(\bar{Y}_r = j)} = q_k/q_j \ . \tag{6.18}$$

<u>Proof</u>. This result follows from the relation

$$P(\bar{Y}_n = k) = u_{n-k}\, q_k \ , \tag{6.19}$$

by an argument very similar to that used to prove the corresponding result for Y_n. The details will therefore be omitted.

Going to the strong version of (6.18), we have the following.

<u>Theorem 6.10</u>. <u>In order that</u>

$$(6.20) \qquad \lim_{n\to\infty} \frac{P(\overline{Y}_{n+m} = k)}{P(\overline{Y}_n = j)} = q_k/q_j ,$$

<u>it is necessary and sufficient that</u>

$$(6.21) \qquad \lim_{n\to\infty} u_n/u_{n-1} = 1 .$$

<u>In order that</u> (6.21) <u>hold it is sufficient that one of the following conditions hold</u>;

(i) e <u>be aperiodic and positive</u>.

(ii) u_n <u>be eventually monotone and</u> e <u>be null</u>.

(iii) <u>condition</u> (5.4) <u>be satisfied</u>.

<u>Proof</u>. (6.20) holds if and only if

$$(6.22) \qquad \lim_{n\to\infty} \frac{P(\overline{Y}_n = k)}{P(\overline{Y}_n = 0)} = q_k .$$

However, it is apparent from (6.19) that (6.22) holds if and only if (6.21) does. That (6.21) holds under condition (i) is a direct consequence of the renewal theorem (see [3], Sec. 6), while the validity of (6.21) under (iii) is obvious. It remains then only to demonstrate that (ii) implies (6.21). To show that this is true, observe first that by assumption, there is an n_o such that $n \geq n_o$ implies u_n

is nonincreasing. If e is null, then $u_n > 0$ for $n > n_o$, for if $u_m = 0$ for $m > n_o$, then $u_n = 0$ for all $n \geq m$, and thus $\Sigma\, u_n < \infty$, which is a contradiction. Thus if $n \geq n_o$, we have $u_{n+1} \leq u_n$ and thus

$$(6.23) \qquad u_{n+1}/u_n \leq 1.$$

On the other hand, if $n > n_o$ we have

$$u_{n+1} = \sum_{k=0}^{n} P(W_1 = n-k)u_k \geq \sum_{k=n_o}^{n} P(W > n-k)u_k$$

$$\geq u_n\, P(W_1 \leq n+1 - n_o)$$

and thus

$$(6.24) \qquad \liminf u_{n+1}/u_n \geq 1.$$

Equation (6.21) now follows from (6.23) and (6.24).

<u>Corollary 6.11</u>. <u>If condition</u> (5.4) <u>holds, then</u>

$$(6.25) \quad \lim_{n\to\infty} P(\overline{Y}_n = k)\ (n-k)^{1-\beta}\, k^{\beta} = \frac{\sin \pi \beta}{\pi} + o(1),$$

<u>where</u> o(1) <u>converges to zero uniformly in</u> n,k <u>as</u> $\min(k, n-k) \to \infty$.

Proof. The result follows directly from Theorem 6.7 and the definition of $\overline{Y}_n$.

We shall conclude this section with the investigation of V_n. As with the preceding quantities, we first have the following general result.

Theorem 6.12. For any recurrent event we have

$$\lim_{n\to\infty} \frac{\sum_{r=0}^{n+m} P(V_r = k)}{\sum_{r=0}^{n} P(V_r = j)} = q_k/q_j \;. \tag{6.26}$$

Proof. It suffices to establish (6.26) for the case of $j = 0$. From the relation

$$P(\overline{V}_n > k,\, \overline{Y}_n > j) \;=\; P(\overline{Y}_{n+k} > k+j),$$

for $k > 0$ we obtain

$$P(V_n = k) = \sum_{r=0}^{n-1} u_r\, P(W_1 = n+k-r).$$

Using the fact that

$$u_{n+r}\,(u_1 + \dots + u_n)^{-1} \to 0 \quad \text{as } n\to\infty,$$

we obtain for $k > 0$

$$(6.27)\quad \lim_{n\to\infty} \frac{\sum_{r=0}^{n+m} P(V_r = k)}{\sum_{r=0}^{n} P(V_r = 0)} = \lim_{n\to\infty} \frac{\sum_{r=0}^{n+m} u_r\{P(W_1 \le n+m+k+1-r)-P(W_1 \le k)\}}{u_o + \dots + u_n}$$

$$= \lim_{n\to\infty} \{P(W_1 \le n+m+1+k-r) - P(W_1 \le k)\} = q_k \ .$$

On the other hand, if $k = 0$, then

$$(6.28)\quad \lim_{n\to\infty} \frac{\sum_{r=0}^{n+m} P(V_r = 0)}{\sum_{r=0}^{n} P(V_r = 0)} = 1 = q_o \ .$$

Equations (6.28) and (6.27) together establish the special case of (6.26) for $j = 0$.

For the strong version of Theorem 6.12 we have the following.

<u>Theorem 6.13</u>. <u>If</u> e <u>is aperiodic and positive, then</u>

$$(6.29)\quad \lim_{n\to\infty} P(V_n = k) = 1/EW_1\, q_k$$

<u>and thus</u>

$$(6.30)\quad \lim_{n\to\infty} \frac{P(V_{n+m} = k)}{P(V_n = j)} = q_k/q_j \ .$$

<u>On the other hand, if condition</u> (5.4) <u>holds, then</u>

$$P(V_n = k) \sim q_k\, n^{\beta-1}/\Gamma(\beta)\, L^{-1}(n), \tag{6.31}$$

<u>and thus</u> (6.30) <u>is valid in this case as well</u>.

<u>Proof</u>. We have

$$P(V_n = k) = P(\overline{Y}_{n+k-1} > k-1) - P(\overline{Y}_{n+k} > k)$$

and

$$P(\overline{Y}_{n+k} = r) = u_{n+k-r}\, q_r .$$

Thus,

$$P(V_n = k) = \sum_{j=0}^{k} u_{n+k-j}\, q_j = \sum_{j=0}^{k-1} u_{n+k-1-j}\, q_j\ . \tag{6.32}$$

Now, if e is aperiodic and positive, then the renewal theorem assures us that

$$u_n \to 1/EW_1 \text{ as } n\to\infty ,$$

and thus (6.29) follows from (6.32) upon taking limits. On the other hand, if condition (5.4) is satisfied, then

$$u_n \sim \frac{n^{\beta-1}\, L^{-1}(n)}{\Gamma(\beta)}$$

and (6.31) now follows from (6.32).

7. A STRONG LIMIT THEOREM

If e is a positive-recurrent event, then $N_n/EN_n \to 1$ with probability one as $n\to\infty$. On the other hand, if e is null recurrent, then Sec. 3 shows that under very wide conditions N_n/EN_n has a nondegenerate limit distribution and so N_n/EN_n will not converge to 1 in probability. The theorem presented below shows that in a certain sense N_n does behave like EN_n. For the special recurrent event, "return to zero," in a one-dimensional random walk, it becomes Theorem 6 in [2]. The proof in the general case is almost the same as in the particular case in [2], and is presented only for completeness.

Theorem 7.1. *Let* $U_n = 1$ *if the null-recurrent event* e *takes place at time* n, *and let* $U_n = 0$ *if not. Take* $\Gamma_n = EN_n$ *and* $u_n = EU_n$. *Then*

$$(7.1) \qquad \lim_{n\to\infty} (\ln \Gamma_n)^{-1} \sum_{k=1}^{n} u_k/\Gamma_k = 1$$

with probability one.

Remark. The above result perhaps gains in interest by comparison with

$$(7.2) \qquad (\ln N_n)^{-1} \sum_{k=1}^{n} U_k/N_k \to 1, \text{ with probability one.}$$

$$(7.3)\qquad (\ln \Gamma_n)^{-1} \sum_{k=1}^{n} u_k/\Gamma_k \to 1 ,$$

both of which are a direct consequence of the Abel-Dini theorem.

Proof. As (7.3) shows,

$$(7.4)\qquad E \sum_{k=1}^{n} U_k/\Gamma_k = \ln \Gamma_n + o(\ln \Gamma_n),$$

$$(7.5)\quad E(\sum_{k=1}^{n} U_k/\Gamma_k)^2 = \sum_{k=1}^{n} u_k/\Gamma_k{}^2 + 2 \sum_{j=1}^{n} u_j/\Gamma_j \sum_{k=j+1}^{n} u_{k-j}/\Gamma_k .$$

Hence,

$$(7.6)\quad \mathrm{var}\left\{\sum_{k=1}^{n} u_k/\Gamma_k\right\} = \sum_{k=1}^{n} \frac{u_k(1-u_k)}{\Gamma_k{}^2} + 2 \sum_{j=1}^{n} u_j/\Gamma_j \sum_{k=j+1}^{n} \frac{u_{k-j}-u_k}{\Gamma_k} .$$

Now, if $n-j \leq j+1$, then

$$\sum_{k=j+1}^{n} \frac{u_{k-j} - u_k}{\Gamma_k} \leq 1/\Gamma_{j+1}\, \Gamma_{n-j} \leq \Gamma_{j+1}/\Gamma_{j+1} = 1,$$

while if $n-j > j+1$,

$$\sum_{k=j+1}^{n} \frac{u_{k-j} - u_k}{\Gamma_k} \leq 1/\Gamma_{j+1} \sum_{k=1}^{j+1} u_k = 1,$$

for $j \geq j_o$, where j_o is such that $\Gamma_{jo} \geq 1$. Hence,

$$\operatorname{var}\left\{\sum_{k=1}^{n} U_k/\Gamma_k\right\} = O(\ln \Gamma_n) .$$

Now, Chebyshev's inequality says that for any $\epsilon > 0$,

(7.7) $P\{|\Sigma U_k/\Gamma_k - \ln \Gamma_n| > \epsilon \ln \Gamma_n\} = O(\ln \Gamma_n)^{-1}$.

Since e is null recurrent, $\Gamma_n/e^{n^2} \to 0$ as $n\to\infty$, and so we may choose a subsequence n_k such that

$$\Gamma_{n_k} \sim e^{k^2} .$$

Thus,

(7.8) $$(\ln \Gamma_{n_k})^{-1} \sum_{j=1}^{n_k} U_j/\Gamma_j \to 1 \text{ as } k\to\infty ,$$

by the Borel-Cantelli Lemma.

To finish up the proof, we next observe that if $n_k \leq n \leq n_{k+1}$, then

$$(7.9)\quad (\ln \Gamma_{n_{k+1}})^{-1} \sum_{k=1}^{n_k} U_j/\Gamma_j$$

$$\leq (\ln \Gamma_n)^{-1} \sum_{k=1}^{n} U_j/\Gamma_j \leq (\ln \Gamma_{n_k})^{-1} \sum_{k=1}^{n_{k+1}} U_k/\Gamma_k .$$

But $\Gamma_{n_{k+1}}/\Gamma_{n_k} \sim \left(\frac{k+1}{k}\right)^2 \sim 1$ as $k \to \infty$, and so (7.1) follows from (7.8) and (7.9).

If e has waiting times which satisfy condition (4.3), then the theorem can be improved thus:

Corollary 7.1. Suppose condition (4.3) is satisfied. Then

$$(7.10)\quad (\beta \ln n)^{-1} \sum_{k=1}^{n} \frac{U_k}{n^\beta/L(n)} \to 1 \text{ with probability one.}$$

Proof. Condition (4.3) is equivalent to $EN_n \sim n^\beta/L(n)$. Now $\ln L(n) = o(\ln n)$, $n \to \infty$. To see this, we note that any slowly varying function can be written in the form

$$L(n) = a(n) \exp \int_1^n \epsilon(t)/t \, dt,$$

where $a(n) \to c > 0$ and $\epsilon(n) \to 0$ as $n \to \infty$. (See Karamata [11]). From this, the above corollary is immediate. Thus

$\ln E N_n \sim \beta \ln n$ as $n \to \infty$, and (7.10) follows.

There are several restatements of (5.1) which are interesting, and we gather these together in the following corollary.

Corollary 7.2. *Let* $EN_n = g(n)$. *Then*

$$(7.11) \quad (\ln g(n))^{-1} \sum_{k=1}^{N_n} g(W_1 + \dots + W_k)^{-1} \to 1 \text{ with probability one,}$$

$$(7.12) \quad \ln g(W_1 + \dots + W_r)^{-1} \sum_{k=1}^{r} g(W_1 + \dots + W_k)^{-1} \to 1$$

with probability one, and if condition (4.3) *is satisfied, then*

$$(7.13) \quad (\beta \ln n)^{-1} \sum_{k=1}^{N_n} (W_1 + \dots + W_k)^{-\beta} L(W_1 + \dots + W_k) \to 1$$

with probability one,

$$(7.14) \quad \beta \ln (W_1 + \dots + W_r)^{-1} \sum_{k=1}^{r} (W_1 + \dots + W_k)^{-\beta} L(W_1 + \dots + W_k) \to 1$$

Proof. (7.11) follows from (7.1) by definition of U_n. (7.12) follows from (7.11) by choosing $n = W_1 + \dots + W_r$. (7.13) and (7.14) follow similarly from (7.10).

8. APPLICATIONS TO SUMS

Let $S_0 = 0$, and $S_n = X_1 + \dots + X_n$ be the n-th partial sum of a sequence $\{X_n\}$ of independent and identically distributed random variables.

Definition. A nonnegative integer n is called a positive (or a strictly positive) ladder point if

$$S_n \geq S_i, \quad 0 \leq i < n, \quad (\text{or } S_n > S_i, \quad 0 \leq i < n),$$

respectively. A similar definition holds for negative and strictly negative ladder points. The fundamental fact about ladder points is that they are recurrent events. From now on we shall denote the strictly positive ladder points by e' and the negative ladder points by e". The quantities defined in Sec. 2, when referred to these events, will be denoted by a prime (e.g., N_n') and a double prime (e.g., W_k''), respectively. Thus, for example, N_n' denotes the number of strictly positive ladder points by time n, and $\{W_k''\}$ the waiting times of negative ladder points, etc.

In terms of the $\{W_k'\}$, we define a sequence Z_n as follows. If $W_1' < \infty$, then define $Z_1 = S_{W_1'}$, while if $W_1' = \infty$, leave Z_1 undefined. Suppose that $Z_1, Z_2 \dots Z_n$ have been defined. Then define $Z_{n+1} = S_{W_1' + \dots + W_{n+1}'} - (Z_1 + \dots + Z_n)$ if $W_{n+1}' < \infty$, and leave Z_{n+1} undefined if $W_{n+1}' = \infty$. Then

(see [14]) the $\{Z_k, W_k'\}$ are independent and identically distributed bivectors. If $M_n = \max(0, S_1, S_2 \dots, S_n)$ and L_n is the time at which M_n first occurs, then

$$(8.1) \qquad M_n = Z_1 + Z_2 + \dots + Z_{N_n'} ,$$

$$(8.2) \qquad L_n = Y_n' .$$

A fundamental theorem due to E. S. Anderson [1] is the following.

<u>Theorem 8.1.</u> <u>(Equivalence Principle)</u> <u>If Q_n and Q_n' are respectively the number of positive and nonpositive sums amongst $S_1, S_2 \dots, S_n$, then $Y_n' = L_n$ and Q_n have the same distribution, and Y_n'' and Q_n' have the same distribution</u>.

A simple proof of Theorem 8.1 can be found in [14]. From this theorem it follows at once that

$$(8.3) \qquad EL_n = EQ_n = \sum_{k=1}^{n} P(S_k > 0),$$

$$(8.4) \qquad EY_n'' = EQ_n' = \sum_{k=1}^{n} P(S_k \leq 0),$$

and thus the results of Sec. 3 yield at once the following theorem.

<u>Theorem 8.2.</u> e' <u>is transient if and only if</u> e" <u>is positive recurrent and</u>

$$(8.5) \qquad 1 - e' = (EW'')^{-1} = \exp(-\sum_{k=1}^{\infty} P(S_k > 0)/k \ .$$

e' <u>is positive recurrent if and only if</u> e' <u>is transient and</u>

$$(8.6) \qquad EW' = (1-e'')^{-1} = \exp \sum_{k=1}^{\infty} P(S_k \leq 0)/k \ .$$

(In (8.5) the

$$\exp(-\sum_{k=1}^{\infty} P(S_k > 0)/k$$

is taken to be zero if the series diverges. In (8.6) both sides are finite or infinite together, and finite if and only if

$$\sum_{k=1}^{\infty} P(S_k \leq 0)/k < \infty \ .)$$

If

$$(8.7) \qquad \lim_{n\to\infty} 1/n \sum_{k=1}^{n} P(S_k > 0)/k = p, \quad 0 \leq p \leq 1 \ ,$$

then

$$(8.8) \qquad (1 - E\, t^{W_1'}) = (1-t)^p \exp \sum_{k=1}^{\infty} t^k (p - P(S_k > 0))/k \,.$$

Furthermore, since $P(Y_n' = n) = P(Q_n = n) =$ $P(S_1 > 0,\ S_2 > 0,\ \ldots,\ S_n > 0)$, a monotone function of n, then conditions (4.3) and (5.4) are equivalent, and

$$(8.9) \qquad P(L_n = n) = P(Y_n' = n) \sim n^{p-1}/\Gamma(p)\ L'(n)^{-1} \,,$$

where

$$(8.10) \qquad L'(n) = \exp \sum_{k=1}^{\infty} \left(1 - \frac{1}{n}\right)^k [p - P(S_k > 0)] \,.$$

<u>Remark</u>. If, in particular the sums S_n are attracted to a stable law with log characteristic function

$$(8.11) \qquad - C\,|\lambda|^{\alpha} (1 + i\,\beta \operatorname{sgn} \lambda\ \omega(\lambda,\alpha)),$$

where $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $C > 0$ and

$$\omega(\lambda,\alpha) = \begin{cases} \tan \frac{\pi\alpha}{2} & \text{if } \alpha \neq 1, \\ \frac{2}{\pi \ln |\lambda|} & \text{if } \alpha = 1, \end{cases}$$

then (8.7) holds by virtue of the fact that $\lim_{n\to\infty} P(S_n > 0) = p$. Moreover, in this case we have

$$(8.12) \qquad p = \frac{1}{2} + \frac{1}{\pi\alpha} \int_0^\infty e^{-C\xi} \sin(-C\beta\xi\ \omega(\alpha,\xi)),$$

which for $\alpha \neq 1$ can be explicitly evaluated* as

$$(8.13) \qquad p = \frac{1}{2} + \frac{1}{\pi\alpha} \arctan\left(-\beta \tan \frac{\pi}{2} \alpha\right).$$

To establish (8.12) let $f(x)$ be the density of the stable law in question. Then formally we have

$$- 1 + 2p = \int_{-\infty}^{\infty} \operatorname{sgn} x\, f(x)dx = \frac{2}{\pi} \int_{-\infty}^{\infty} f(x)dx \int_0^\infty \frac{\sin \xi x}{\xi} d\xi$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\sin \xi x}{\xi} d\xi \int_{-\infty}^{\infty} f(x)dx$$

$$= \frac{2}{\pi} \int_0^\infty \frac{e^{-C\xi^\alpha}}{\xi} \sin(-C\beta\xi^\alpha\ \omega(\xi,\alpha))d\xi$$

$$= \frac{2}{\pi\alpha} \int_0^\infty e^{-C\xi} \sin(-C\beta\xi\ \omega(\xi,\alpha))d\xi,$$

which for $\alpha \neq 1$ is

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*We are indebted to O. Gross for showing that the formal evaluation of p could be performed by the formal calculation given here.

$$\frac{2}{\pi\alpha}\int_0^\infty e^{-C\xi}\sin(-C\beta\xi\tan\frac{\pi\alpha}{2}) = \frac{2}{\pi\alpha}\arctan(-\beta\tan\frac{\pi\alpha}{2}).$$

(See table of Laplace transforms on page 125 of [13]). To make this formal derivation rigorous, we must justify the interchange in the order of integration used above. This follows from the standard theorems (see e.g. Hobson's Treatise of Real Variables, p. 349, Vol. 2 of the Dover Edition) by observing that

(a) $\int_0^a \int_\alpha^\beta \left| \frac{\sin \xi x}{\xi} f(x) \right| d\xi\, dx < \infty$ for all finite a, α, β.

(b) Each of the required iterated integrals exist.

(c) $\left| \int_0^\mu \frac{\sin \xi x}{\xi} d\xi \left(\int_{-\infty}^a + \int_b^\infty\right) f(x)dx \right| < \mu \left(\int_{-\infty}^a + \int_b^\infty\right) f(x)dx$

$\to 0$ as $a \to -\infty$, $b \to \infty$.

(d) $\left| \int_{-\infty}^\infty f(x) \int_b^\infty \frac{\sin \xi x}{\xi} d\xi \right| \le C/b \to 0,\ b \to \infty$,

where C is a positive constant.

For the special case when $EX_1 = 0$, var $X_1 = \sigma^2 < \infty$ ($\alpha = 2$), Spitzer in [16] has shown the following important result.

<u>Theorem 8.3</u>. <u>If</u> $EX_1 = 0$, var $X_1 = \sigma^2 < \infty$, <u>then the series</u>

$$\sum_{k=1}^{\infty} 1/k \, (1/2 - P(S_k > 0))$$

<u>converges, and if</u> γ <u>is its sum, then</u>

(8.14) $$\lim_{n\to\infty} L'(n) = e^{\gamma} .$$

<u>Furthermore, in this case</u> $EZ_1 < \infty$ <u>and</u>

(8.15) $$EZ_1 = \sigma/\sqrt{2}\, e^{\gamma} .$$

From (8.14) we have that (8.9) improves to

(8.16) $$P(L_n = n) = P(Q_n = n) \sim e^{-\gamma} (n\pi)^{-1/2}, \; n\to\infty ,$$

as was found by Spitzer in [16]. Use of the above facts enables us to translate all of the results of Secs 3-5 into results on the functionals associated with e'. As examples we have the following.

<u>Theorem 8.4.</u> <u>Suppose</u> $EX_1 = 0$, var $X_1 = \sigma^2 < \infty$ <u>and</u> $0 < s \leq 1$. <u>Then</u>

(8.17) $$\lim_{n\to\infty} P(M_n/\sigma\sqrt{n} \leq x,\; L_n/n \leq y)$$

$$= 1/\pi \int_0^x \int_0^y x\, e^{-x^2/2y}\, y^{-1}(1-y)^{-1/2}\, dx\, dy,$$

where $x \geq 0$, $0 \leq y \leq 1$,

$$(8.18)\quad \lim_{n\to\infty} P(M_n/\sigma\sqrt{n} \leq x \mid L_n = [sn]) = \begin{cases} 1 - e^{-x^2/2s}, & (x \geq 0), \\ 0, & x < 0 . \end{cases}$$

Proof. The strong law of large numbers asserts that

$$\frac{Z_1 + \dots + Z_n}{n} \to EZ_1$$

with probability one. Thus,

$$\frac{M_n}{N_n} = \frac{Z_1 + \dots + Z_{N_n}}{N_n} \to EZ_1$$

with probability one. In addition, we have

$$b_n = n^{1/2}/L'(n) \sim n^{1/2} e^{-\alpha}, \quad \text{as } n\to\infty ,$$

and consequently

$$EZ_1\, b_n \sim \sigma(n/2)^{1/2} .$$

Thus, by Corollary 4.2,

$$\lim_{n\to\infty} P(M_n/\sigma\sqrt{n} \le x,\ L_n/n \le y)$$

$$= \lim_{n\to\infty} P\left(\frac{M_n}{\sqrt{2}\, N_n} \frac{N_n}{EZb_n} \le x,\ L_n/n \le y\right) = K_{1/2}\,(\sqrt{2}\, x,\ y)\sqrt{2}\ ,$$

where $K_{1/2}\,(x, y)$ is given in formula (4.22). This proves (8.17), and (8.18) follows similarly from Theorem 5.3.

<u>Remark</u>. In the above proofs the only essential use which was made of the assumptions $EX = 0$, $\text{var } X = \sigma^2 < \infty$ was to guarantee that $EZ_1 < \infty$. More generally we have that

$$\lim_{n\to\infty} P\left(\frac{M_n}{EZ_1\, b_n} \le x,\ \frac{L_n}{n} \le y\right) = \int_0^x \int_0^y K_p(x, y),$$

$$\lim_{n\to\infty} P\left(\frac{M_n}{EZ_1\, b_n} \le x \mid L_n = [sn]\right) = \int_0^{xs^{-p}} h_p(x)\, dx\ ,$$

whenever X_1 has a distribution for which $EZ_1 < \infty$ and (8.7) holds. In particular, this will be the case if $EX < 0$ or if (8.7) holds, $EX = 0$, and X_1 is bounded above.

By using Theorem 6.7 and (8.9) we have the following local form of the generalized arc-sin laws for Q_n.

<u>Theorem 8.5</u>.

$$k^{1-\beta}(n-k)^{-\beta}\ P(Q_n = k) = \frac{\sin \pi\beta}{\pi} + o(1),$$

where $o(1)$ converges to zero uniformly in k, n, as $\min(k, n-k) \to \infty$.

For the case when the $\{X_n\}$ are aperiodic* integer-valued random variables, a recurrent event which has been much studied is the event $S_n = 0$. The results of the preceeding sections may be translated to give information about the functionals associated with this event, by use of the following lemma.

Lemma 8.6. If for some slowly varying function $h(n)$ and some α, $0 < \alpha \leq 2$, $S_n/h(n)n^{1/\alpha}$ converges in distribution to the stable law with log characteristic function given in (8.11), then

$$(8.19) \qquad \lim_{n\to\infty} h(n)n^{1/\alpha}\, g(o)\, P(S_n = 0) = 1,$$

where $g(0)^{-1}$ is the value of the density function at 0 of the law given in (8.11).

Moreover, if $\alpha \neq 1$ then

$$(8.20) \quad g(0)^{-1} = \frac{\Gamma(1/\alpha)[(C + i\,\beta \tan \frac{\pi\alpha}{2})^{1/\alpha} + (C - i\,\beta \tan \frac{\pi\alpha}{2})^{1/\alpha}]}{2\pi\alpha\,(C^2 + \beta^2 \tan^2 \frac{\pi\alpha}{2})^{1/\alpha}},$$

while if $\alpha = 1$ we have

$$(8.21) \quad g(0)^{-1} = \frac{1}{2\pi}\int_0^\infty e^{-C\lambda} \cos(\tfrac{2}{\pi}\, C\, \beta\, \lambda\, \ln \lambda)\,d\lambda .$$

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*I.e., $|E\, e^{i\,\theta\, X_1}| = 1$ if and only if $\theta = 2n\pi$ for integer n.

For the symmetric case (when $\beta = 0$) we have

$$(8.22) \qquad g(0)^{-1} = \frac{\Gamma(1/\alpha)}{\pi\alpha\, C^{1/\alpha}} \qquad 0 < \alpha \leq 2 .$$

Proof. Equation (8.19) is an immediate consequence of the local limit theorem for lattice distributions (see Sec. 50 of [10]). To evaluate $g(0)^{-1}$ observe that if $\alpha \neq 1$, then

$$g(0)^{-1} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-C|\lambda|^{\alpha}(1+i\beta \operatorname{sgn}\lambda \tan\frac{\pi\alpha}{2}} d\lambda$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} e^{-C\lambda^{\alpha}} \cos\left(C\lambda^{\alpha}\beta \tan\frac{\pi\alpha}{2}\right) d\lambda$$

$$= \frac{1}{2\pi\alpha}\int_{0}^{\infty} e^{-C\lambda} \cos\left(C\lambda\beta \tan\frac{\pi\alpha}{2}\right) \lambda^{1/\alpha-1}\, d\lambda .$$

From the tables of Laplace transforms in [13] on p. 125, this last integral may be evaluated. This results in the right-hand side of (8.20). A similar calculation gives (8.21).

Remark. From (8.19) it follows at once that if the S_n satisfy Lemma 8.6, then the event, "$S_n = 0$," is transient if $\alpha < 1$ and null recurrent if $\alpha \geq 1$. We may then apply the results of the previous sections to obtain information about this particular recurrent event. As a novel application we have the following.

Theorem 8.7. *Let* $B_i(x)$ *for* $i \le i \le r$ *be* r *functions on the integers such that for each* i,

$$(8.23) \qquad \sum_x |B_i(x)| < \infty \quad \text{and} \quad \Sigma B_i(x) = h_i \neq 0.$$

Then if (8.19) *is satisfied for* $1 \le \alpha \le 2$,

$$(8.24) \quad \lim_{n\to\infty} P\Big(\sum_{k=0}^{n} h_i \,(S_k)/B_i \,\gamma_n \le x_i,\ 1 \le i \le r;\ Y_n/n \le y\Big)$$

$$= \int_0^{\min(x_1,\ldots,x_r)} \int_0^y K_{1-1/\alpha}(x,\, y)\, dx\, dy,$$

$$(8.25) \quad \lim_{n\to\infty} P\Big(\sum_{k=0}^{n} B_i(S_k)/B_i \gamma_n \le x_i,\ 1 \le i \le r \mid Y_n = [ns]\Big)$$

$$= \int_0^{s^{-1+1/\alpha} \min(x_1,\ldots,x_r)} h_{1-1/\alpha}(u)\, du\ ,$$

where

$$\gamma_n = n^{1-1/\alpha}/h(n)\ g(0)^{-1}\ (1-1/\alpha)^{-1}$$

and $K_{1-1/\alpha}(x,\, y)$ *is given in* (4.21), $h_{1-1/\alpha}(x)$ *is given in* (4.18) *and* Y_n *is the time of the last return to zero.*

Proof. As the proof of (8.24) and (8.25) are almost

identical, we shall only prove (8.24). By Corollary 2, Sec. 15 of [3], if N_n is the number of returns to zero during the first n steps, then for each i, $1 \leq i \leq r$,

$$(8.26) \qquad \lim_{n\to\infty} \sum_{k=1}^{n} B_i(S_k)/N_n = B_i$$

with probability one. Equation (8.19) implies that

$$EN_n \sim \gamma_n = n^{1-1/\alpha}/h(n)\ g(0)^{-1}\ (1-1/\alpha)^{-1},$$

and so by Theorem 4.2,

$$(8.27) \quad \lim_{n\to\infty} P(N_n/\gamma_n \leq x,\ Y_n/n \leq y) = \int_0^x \int_0^y K_{1-1/\alpha}(x,\ y)\ dx\ dy.$$

Combining (8.26) and (8.27), we obtain (8.24).

To conclude this section let us note that whenever the S_n satisfy the conditions of Lemma 8.6 for $1 \leq \alpha \leq 2$ we have that condition (6.3) is satisfied. Thus we may apply Corollary 6.3 to obtain the following result.

If N_n is the number of zeros amongst the first n sums S_n which satisfy the conditions of Lemma 8.6, then

$$\lim_{n\to\infty} P(N_n = j \mid N_n \leq k) = (k+1)^{-1} .$$

APPENDIX

We present here the extensions of the Tauberian and Abelian theorems which include Lemmas 5.4 and 5.5 as special cases. The special case of theorem A.1 for $L(x)$ = constant is due to Landau, and may be found on p. 517 of [5] as Helfsatz 3. The Abelian Theorem A.4 can be found for the special case of $L(x)$ = constant as Theorem 41 on p. 98 of Hardy's famous treatise on divergent series.

Theorem A.1. Suppose

$$F(t) = \int_0^t \phi(u)\, du, \tag{A.1}$$

(A.2) $\phi(u)$ is monotone for u sufficiently large,

$$F(t) \sim A t^{\gamma} L(t), \quad t \to \infty, \tag{A.3}$$

where $\gamma > 0$ and $L(x)$ is a slowly varying function (i.e., $L(x) > 0$ and $L(ax) \sim L(x)$, $x \to \infty$, for every positive a). Then

$$\varphi(t) \sim \gamma A t^{\gamma - 1} L(t), \quad t \to \infty. \tag{A.4}$$

Proof. We shall assume that $\varphi(u)$ is positive and monotone increasing if t is sufficiently large. Only obvious modifications are needed to take care of the other possibilities. Thus if $t > T_o$, $\varphi(t)$ is positive and monotone. If $0 < \alpha \leq 1$ and $t > T_o/\alpha$,

$$\varphi(\alpha t)t(1-\alpha) \leq \int_{\alpha t}^{t} \varphi(u)du \leq \varphi(t)t(1-\alpha),$$

and so

$$\phi(\alpha T) \leq \frac{F(t) - F(\alpha t)}{t(1-\alpha)} \leq \varphi(t),$$

$$\varphi(\alpha t)/t^{\gamma-1} L(t) \leq \frac{F(t) - F(\alpha t)}{t^{\gamma} L(t)(1-\alpha)} \leq \frac{\varphi(t)}{t^{\gamma-1} L(t)} .$$

Thus

$$\text{(A.5)} \quad \overline{\lim_{t\to\infty}} \frac{\varphi(\alpha t)}{t^{\gamma-1} L(t)} \leq \frac{A(1-\alpha^{\gamma})}{1-\alpha} \leq \underline{\lim_{t\to\infty}} \varphi(t)/t^{\gamma-1} L(t) .$$

Now

$$\overline{\lim_{t\to\infty}} \frac{\varphi(\alpha t)}{t^{\gamma-1}\, L(t)} = \overline{\lim_{t\to\infty}} \frac{\varphi(\alpha t)}{(\alpha t)^{\gamma-1}\, L(\alpha t)}\, \alpha^{\gamma-1}\, L(t)$$

$$= \overline{\lim_{t\to\infty}} \frac{\varphi(t)}{t^{\gamma-1}\, L(t)}\, \alpha^{\gamma-1} .$$

Thus

$$\alpha^{\gamma-1}\, \overline{\lim_{t\to\infty}} \frac{\varphi(t)}{t^{\gamma-1}\, L(t)} \le \frac{A(1-\alpha^{\gamma})}{1-\alpha} \le \underline{\lim} \frac{\varphi(t)}{t^{\gamma-1}\, L(t)} .$$

Taking the limit as $\alpha\to 1$, we obtain

$$\overline{\lim_{t\to\infty}} \frac{\varphi(t)}{t^{\gamma-1}\, L(t)} \le A\gamma \le \underline{\lim}_{t\to\infty} \frac{\varphi(t)}{t^{\gamma-1}\, L(t)} ,$$

which completes the proof.

<u>Corollary A.2</u>. <u>If</u> (A.1), (A.2), (A.3) <u>hold except that</u> $\gamma = 0$, <u>then we may conclude that</u>

$$\varphi(t) = o(L(t)/t), \quad t\to\infty .$$

<u>Proof</u>. If $\gamma = 0$, then (5) becomes

$$\text{(A.6)} \quad \overline{\lim} \frac{\varphi(\alpha t)}{t^{\gamma-1}\, L(t)} \le 0 \le \underline{\lim}_{t\to\infty} \frac{\varphi(t)}{t^{\gamma-1}\, L(t)} ,$$

and the remainder of the proof is the same.

Corollary A.3. If the sequence a_k, $a_k \geq 0$, is monotone, and

$$a_1 + \dots + a_n \sim n^{\gamma} L(n), \quad n \to \infty,$$

then

$$a_n \sim \gamma\, n^{\gamma-1} L(n). \tag{A.7}$$

Proof. Choose $\varphi(t) = a_n$ if $n-1 \leq t < n$. Then

$$\int_0^n \varphi(t)dt = a_1 + \dots + a_n = F(n),$$

and (A.7) follows from Theorem A.1.

Theorem A.4. If $r > -1$, $s > -1$ and $a(x)$, $b(x)$ are integrable on any finite positive interval, and if

$$a(x) \sim x^r L(x), \quad b(x) \sim x^s h(x),$$

for $L(x)$, $h(x)$ slowly varying functions, then

$$\text{(A.8)}\quad C(x) = \int_0^x a(t)\, b(x-t)dt$$

$$\sim \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\, h(x)\, L(x)\, s^{r+s+1}, \quad x\to\infty .$$

<u>Proof</u>. Suppose $0 < \delta < 1/2$, $\delta^{r+1} < (r+1)\epsilon$, $\delta^{s+1} < (s+1)\epsilon$ and

$$\text{(A.9)}\quad \gamma = \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}$$

$$= \int_0^1 u^r(1-u)^s\, du \;<\; \int_\delta^{1-\delta} u^r(1-u)^s\, du + \epsilon .$$

Write

$$C(x) = \int_0^{\delta x} + \int_{\delta x}^{(1-\delta)x} + \int_{(1-\delta)x}^{x} = C_1(x) + C_2(x) + C_3(x) .$$

By the assumptions, there is some $x_o(\epsilon)$ such that if $x > x_o(\epsilon)$, we have

$$(1-\epsilon)\int_{\delta x}^{(1-\delta)x} u^r(x-u)^s\ L(u)h(x-u)du$$

$$\leq C_2(x) \leq (1+\epsilon)\int_{\delta x}^{(1-\delta)x} u^r(x-u)^s\ L(u)h(x-u)du\ .$$

Thus,

$$\text{(A.10)}\quad (1-\epsilon)\ x^{r+s+1}\ L(x)h(x)\int_{\delta}^{1-\delta} v^r(1-v)^s\ \frac{L(vx)h((1-v)x)}{L(x)h(x)}\,dv \leq C_2(x)$$

$$\leq (1+\epsilon)x^{r+s+1}L(x)h(x)\int_{\delta}^{1-\delta} \frac{L(vx)h(x(1-v))}{L(x)h(x)}\ v^r(1-v)^s\ dv\ .$$

Now, it is a basic fact of slowly varying functions that $L(ax)/L(x)$ converges to 1 as $x\to\infty$ uniformly in a, for a bounded away from zero. (See [11]).

Thus, from (A.10) we obtain

$$\text{(A.11)}\quad \overline{\lim_{x\to\infty}}\ \frac{C_2(x)}{x^{r+s+1}\ L(x)h(x)} \leq (1+\epsilon)\int_{\delta}^{1-\delta} v^r(1-v)^s\ dv \leq (1+\epsilon)\gamma\ ,$$

and

$$\text{(A.12)} \quad \varliminf_{x\to\infty} \frac{C_2(x)}{x^{r+s+1}\,L(x)h(x)} \geq (1-\epsilon)\int_\delta^{1-\delta} u^r(1-u)^s\,du \geq (1-\epsilon)\,\gamma-\epsilon \;.$$

Simple computations show that for some $K > 0$,

$$\text{(A.13)} \quad |C_1(x)| \leq K\epsilon\; L(x)h(x)\; x^{r+s+1} \;,$$

$$\text{(A.14)} \quad |C_3(x)| \leq K\epsilon\; L(x)h(x)\; x^{r+s+1} \;.$$

Combining (A.11), (A.12), (A.13) and (A.14), we obtain (A.8).

Corollary A.5. *If $\{c_n^{(m)}\}$ is the mth-fold convolution of $\{a_n\}$ with itself and*

$$a_n \sim n^{-\alpha}/L(n), \qquad 0 < \alpha < 1 \;,$$

then

$$\text{(A.15)} \quad c_n \sim L(n)^{-m}\, n^{-m\alpha+m-1}\, \Gamma(1-\alpha)^m/\Gamma(m(1-\alpha)) \;.$$

Proof. If we choose $a(x) = a_n$, $n \leq x < n+1$, $b(x) = b_n$, $n \leq x < n+1$, we obtain

$$c_n^{(2)} \sim \frac{\Gamma(1-\alpha)^2}{\Gamma(2(1-\alpha))} L(n)^{-2} n^{-2\alpha+1} ,$$

and (A.15) follows by induction.

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